Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **SEP 2008**	2. REPORT TYPE	3. DATES COVERED **00-00-2008 to 00-00-2008**

4. TITLE AND SUBTITLE **A Full Spectrum Solution to Wave Propagation Prediction**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Naval Undersea Warfare Center, ,Newport,RI,02841**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM002176. Presented at the MTS/IEEE Oceans 2008 Conference and Exhibition held in Quebec City, Canada on 15-18 September 2008. U.S. Government or Federal Rights License.

14. ABSTRACT
see report

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**3**	

A Full Spectrum Solution to Wave Propagation Prediction

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***Abstract*— An overview of a normal mode method of solving the Helmholtz wave equation to describe the underwater sound field for a fixed point source in a plane multilayered medium is presented. The mode functions are well-defined at all depths of the medium as they are continuous across turning points of the separated depth-dependent differential equation. Comparisons of model results to a limited number of benchmark propagation soultions are presented.**

I. Introduction

This paper presents a normal mode propagation model which utilizes Bessel functions of order $1/3$ as solutions to the depth dependent wave equation. The use of $1/3$-order Bessel functions to predict propagation is not new (see, for example [1]). The advantage of the approach is that it marries the computational efficiency of asymptotic solutions [2] with the rigor of normal mode solutions. Unlike traditional asymptotic methods (see, for example [3]), the Bessel function formulation enables the normal mode amplitude functions to be computed so as to be continuous at all depths of a stratified ocean, and in particular to behave linearly through the turning points of the separated differential equation.

A. Model Approach

Assuming sound pressure φ has no azimuthal dependence, the Hemholtz wave equation can be expressed in cylindrical coordinates as

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2},$$

where r represents horizontal range, t represents time, c is sound speed as a function of depth, z, which is increasing down from the ocean surface. After separation of variables, the depth dependent function $u(z)$ is given by

$$u''(z) + k_z^2(z)u(z) = \delta(z-z_s), \tag{1}$$

where the separation constant k_r corresponds to the horizontal component of the local wavenumber, $k_z^2(z) = \frac{\omega^2}{c^2(z)} - k_r^2$ is the vertical component of the local wavenumber, and we assume continuous propagation of a harmonic point source of unit intensity at depth z_s.

Dropping the subscript r on the horizontal wavenumber k_r for simplicity of notation, let $z = z_0$ be a turning point for the wavenumber k, thus $k_z^2(z_0,k) = 0$. By expressing k_z^2 as a linear function of z in a neighborhood z_0 and performing a change of variable to express Eq. (1) as a function of vertical phase, $\xi(z,k) = \int_{z_0}^{z} k_z(z,k)dz$, yields Bessel's equation of order $1/3$ [4]

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + (1 - \frac{1}{9\xi^2}) = 0.$$

Thus cylinder functions of order $1/3$ provide solutions for u, for example,

$$u = (z-z_0)^{1/2}J_{1/3}(\xi).$$

We choose, for convenience, to use the modified Bessel functions $\xi^{1/3}J_{1/3}(\xi)$ as the fundamental solutions for the Green's function construction. Although the Bessel functions are used for computation, the Hankel functions are better suited to the acoustic problem. For consistency with the asymptotic exponential solutions used in the Integrated Mode approach [5], we express the Hankel functions as linear combinations of Bessel functions,

$$H_{1/3}^{(1)}(\xi) = \tfrac{ie^{-i\pi/3}}{\sqrt{3}/2}J_{1/3}(\xi) - \tfrac{i}{\sqrt{3}/2}J_{-1/3}(\xi),$$

$$H_{1/3}^{(2)}(\xi) = \tfrac{-ie^{-i\pi/3}}{\sqrt{3}/2}J_{1/3}(\xi) + \tfrac{i}{\sqrt{3}/2}J_{-1/3}(\xi).$$

Neglecting higher order terms, since ξ and k_z^2 are proportional to $(z-z_0)^{3/2}$ and $(z-z_0)$, respectively, we have $(z-z_0)$ proportional to ξk_z^{-1}. Thus, for the case of linear turning points, a set of allowable solutions for propagation towards and away from z_0 when compared with time dependence $e^{i\omega t}$ are given by

$$\begin{aligned} u_1 &= K_1\sqrt{\tfrac{\xi}{k_z}}H_{1/3}^{(1)}(\xi), \\ u_2 &= K_2\sqrt{\tfrac{\xi}{k_z}}H_{1/3}^{(2)}(\xi), \end{aligned} \tag{2}$$

where $k_z^2 > 0$ is assumed and $K_{1,2} = \sqrt{\frac{\pi}{2}}e^{\pm\frac{5\pi}{12}}$, respectively.

The reason for using the Hankel functions is apparent when the asymptotic expansions are employed. The leading terms of the asymptotic expansions of $H_{1/3}^{(1)}(\xi)$ and $H_{1/3}^{(2)}(\xi)$ when $arg(\xi) > -\pi$ are given by [4] as

$$\begin{aligned} H_{1/3}^{(1)}(\xi) &\sim \sqrt{\tfrac{2}{\pi\xi}}e^{i(\xi-5\pi/12)}, \\ H_{1/3}^{(2)}(\xi) &\sim \sqrt{\tfrac{2}{\pi\xi}}e^{-i(\xi-5\pi/12)}. \end{aligned} \tag{3}$$

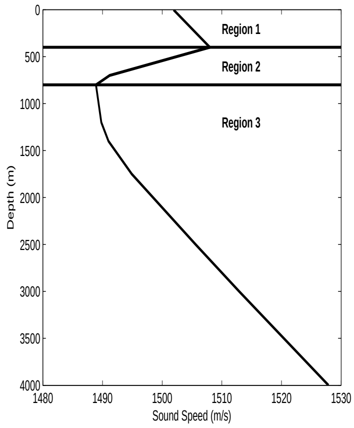

Fig. 1. Indexing of Sound Speed Regions with Monotone Gradient

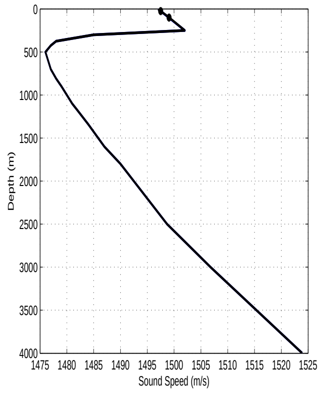

Fig. 2. Sound Speed Profile for PE Workshop Test Case 7

Substitution of (3) into (2) yields

$$u_1 \sim \sqrt{\frac{2}{\pi}} e^{-i5\pi/12} \frac{1}{\sqrt{k_z}} e^{i\xi},$$

$$u_2 \sim \sqrt{\frac{2}{\pi}} e^{-i\pi/12} \frac{1}{\sqrt{k_z}} e^{-i\xi},$$

which are valid away from the turning point $z = z_0$ on the side where $|cos\theta| < 1$ (they become infinite when $k_z^2 = 0$). The Bessel function expressions (2) allow the solution of the differential equation to be approximated over a specific z-interval, both near and away from z_0, and they agree with the exponential solutions (3) in regions where they are oscillatory.

1) Solutions in Layered Media: For the acoustic propagation problem, the modified Hankel functions (2) cannot be used in a depth interval where more than one turning point exists, as in a refracting sound channel, for example. Thus, we subdivide the water column into regions containing at most one turning point as shown in Fig. 1.

The fundamental solutions are computed in each region by measuring the vertical phase to the only possible turning point. A solution satisfying a given boundary condition is then expressible as a linear combination of the two fundamental solutions. If the coefficients of the required linear combination are known in one region for a given wavenumber, k, the solution can be extended into an adjacent region by satisfying continuity conditions for solutions and their derivatives across the region boundary.

2) Benchmark Cases: A number of benchmark cases are presented for which the model results agree with KRAKEN [6] and COUPLE [7]. The first case is Test Case 7 from the Second Parabolic Equation Workshop held at Stennis Space Center, MS in May, 1991 [8], and the remaining three cases are taken from a text by C. Allan Boyles [9]. Since the COUPLE model results are in agreement with KRAKEN in all cases, only the KRAKEN results are included in the plots.

II. Conclusion

A normal mode model which utilizes Bessel functions of order 1/3 has been verified on a number of benchmark cases

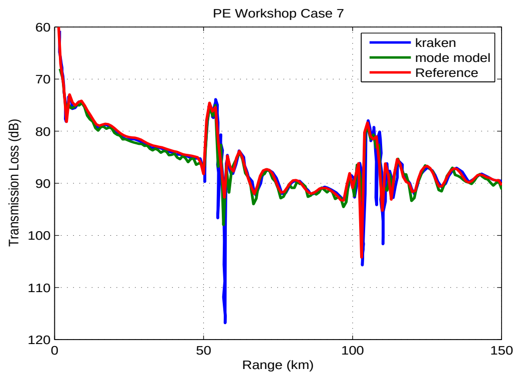

Fig. 3. Propagation Loss for PE Workshop Test Case 7

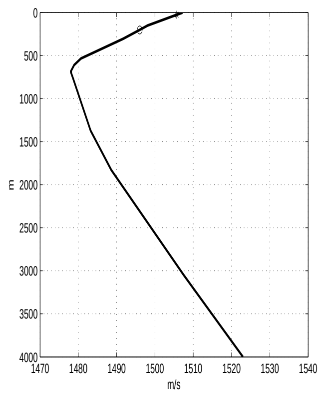

Fig. 4. Sound Speed Profile for Boyles Convergence Zone Environment

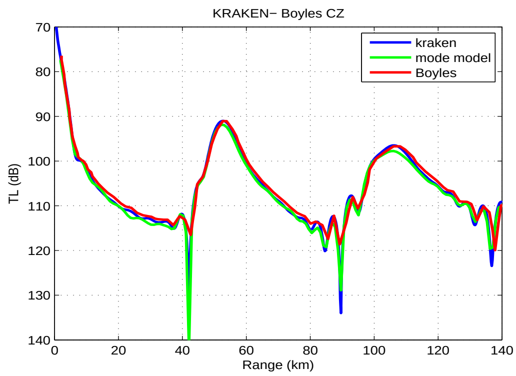

Fig. 5. Propagation Loss for Boyles Convergence Zone Environment

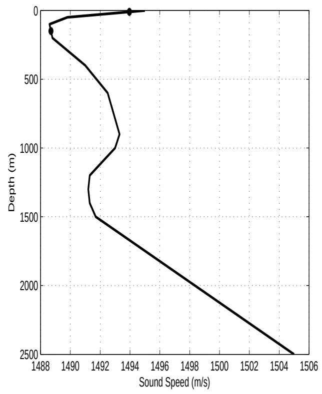

Fig. 6. Sound Speed Profile for Boyles Double Duct Environment

and one measured data set, all involving water-borne propagation. An overview of the model derivation was followed by a set of model predictions for modes within the water column. The solution represents a compromise between traditional asymptotic solutions which are computationally efficient and more rigorous models which are typically computation-intensive. Thus, the rigor of more comprehensive solutions is retained without sacrificing the efficiency of asymptotic methods. This is accomplished by a particular formulation of the characteristic functions using Bessel functions of order $1/3$, thus enabling the mode amplitude functions to remain continuous through turning points of the depth-dependent differential wave equation. The Bessel function argument is chosen carefully in various regions of the complex plane so as to avoid the Stokes phenomenon where solutions can change abruptly across region boundaries.

The long-term goal of this research is the development, implementation, and fielding of a concise but rigorous full-spectrum prediction capability which integrates the above approach with solutions which handle both boundary interactions and mode coupling effects which occur in the propagation of low frequency signals in shallow water environments. The integration of these methods is full-spectrum in the sense that it is contiuous across all operational sonar frequencies and in

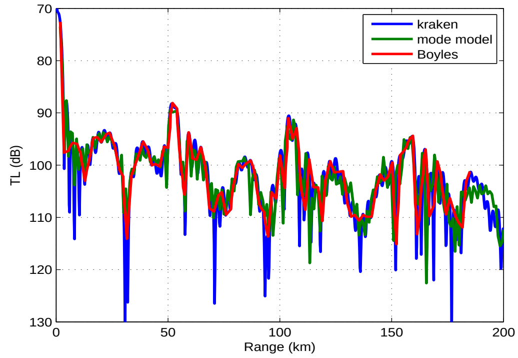

Fig. 7. Propagation Loss for Boyles Double Duct Environment

transitions between deep and shallow water.

ACKNOWLEDGMENT

The authors wish to acknowledge Dr. G. A. Leibiger, formerly of NUWC, in partnership with whom the approach documented herein was developed.

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